RIEMANN HYPOTHESIS: A SPECIAL CASE OF THE RIESZ AND HARDY-LITTLEWOOD WAVE AND A NUMERICAL TREATMENT OF THE BAEZ-DUARTE COEFFICIENTS UP TO SOME BILLIONS IN THE K-VARIABLE

STEFANO BELTRAMINELLI AND DANILO MERLINI

Abstract. We consider the Riesz and Hardy-Littlewood wave i.e. a "critical function" whose behaviour is concerned with the possible truth of the Riemann Hypothesis (RH). The function is studied numerically for the case $\alpha = \frac{15}{2}$ and $\beta = 4$ in some range of the critical strip, using Maple 10.

In the experiments, $N = 2000$ is the maximum argument used in the Möbius function appearing in c_k i.e. the coefficients of Baez-Duarte, in the representation of the inverse of the Zeta function by means of the Pochammer's polynomials.

The numerical results give some evidence that the critical function is bounded for $\Re(s) > \frac{1}{2}$ and such an "evidence" is stronger in the region $\Re(s) > \frac{3}{4}$ where the wave seems to decay slowly. This give further support in favour of the absence of zeros of the Riemann Zeta function in some regions of the critical strip ($\Re(s) > \frac{3}{4}$) and a (weaker) support in the direction to believe that the RH may be true ($\Re(s) > \frac{1}{2}$).

1. Introduction

The starting point of this note is the representation of the reciprocal of the Riemann Zeta function by means of the Pochammer's polynomials $P_k(z)$ (where z is a complex variable), whose coefficients c_k have been introduced by Baez-Duarte for the Riesz case ($\alpha = \beta = 2$). For the study of the coefficients c_k, some recent analytical as well as numerical results have been obtained [2, 3, 4, 5, 6, 7, 8, 9]. For a rigorous treatment of the Müntz formula to the finding of new zero free regions of the Riemann Zeta function, the reader may consult the work of Albeverio and Cebulla [1].

Using the Baez-Duarte approach, the representation of $\frac{1}{\zeta(s)}$ may be obtained for a family of a two parameter Pochammer's polynomials (parameters α and β [4]) and reads:

$$\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k(\alpha,\beta) P_k(s,\alpha,\beta), \tag{1.1}$$

where

$$P_k(s,\alpha,\beta) \quad := \prod_{r=1}^{k} \left(1 - \frac{\frac{s-\alpha}{\beta}+1}{r}\right) \tag{1.2}$$

Date: 17 September 2006.

1991 *Mathematics Subject Classification.* 11M26.

Key words and phrases. Riemann Zeta function, Riemann Hypothesis, Criteria of Riesz, Hardy-Littlewood and Baez-Duarte, Pochammer's polynomials.

$$c_k(\alpha,\beta) \quad := \sum_{n=1}^{\infty} \tfrac{\mu(n)}{n^\alpha}\left(1-\tfrac{1}{n^\beta}\right)^k. \tag{1.3}$$

The expression for c_k we will use in our computation is given by:

$$c_k(\alpha,\beta) = \sum_{n=1}^{N} \frac{\mu(n)}{n^\alpha} e^{-\frac{k}{n^\beta}}, \tag{1.4}$$

which for large k is a correct formula for the c_k given above (see Appendix). From a theorem of Baez-Duarte [2, 3], an important inequality concerning the Pochammer's polynomials is given by:

$$|P_k(z)| \leq Ak^{-\Re(z)}. \tag{1.5}$$

The inequality, when applied to our family of Pochammer's polynomials gives:

$$\left|P_k(\frac{s-\alpha}{\beta}+1)\right| \leq Ak^{-(\frac{\Re(s)-\alpha}{\beta}+1)}. \tag{1.6}$$

From this it follows [2, 3, 4] that the RH will be true, i.e. that $\frac{1}{\zeta(s)}$ in the representation above will be different from infinity (no zero of $\zeta(s)$ for $\Re(s) > \rho$) if $c_k k^{\frac{\alpha-\rho}{\beta}} \leq C$. For the numerical study it is convenient to introduce the variable $x = \log(k)$, in term of which we define the critical function corresponding to α and β. This is given by:

$$\psi(x;\alpha,\beta,\rho) := e^{\frac{\alpha-\rho}{\beta}x} \sum_{n=1}^{2000} \frac{\mu(n)}{n^\alpha} e^{-\frac{e^x}{n^\beta}}. \tag{1.7}$$

2000 is the maximum argument N used in these experiments, which for the special case we treat ($\alpha = \frac{15}{2}$ and $\beta = 4$) ψ will be calculated up to $x = 30$ (this corresponds to $k = e^{30} = 1.06865 \times 10^{13}$).

Before we present the results of our numerical experiments for various values of ρ (for $\rho = 1, \frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{3}{10}$) it is important to give the explicit expression of the contribution of the non trivial (ψ_{nt}) and also of the trivial zeros (ψ_t) to the critical function defined above for the general case α and β, following the expression given by Baez-Duarte for the case $\alpha = \beta = 2$ [2]. For the non trivial zeros, in the variable $x = \log(k)$, at ρ, it is given by:

$$\psi_{nt}(x;\alpha,\beta,\rho) = \frac{1}{\beta}\sum_{z} \frac{e^{\frac{i\Im(z)}{\beta}x}\Gamma(-\frac{\Re(z)+i\Im(z)-\alpha}{\beta})}{\zeta'(z)}, \tag{1.8}$$

where z is any nontrivial zero of $\zeta(s)$. In our experiments we will limit to the contribution of the first two lower zeros given experimentally by $z_1 = \frac{1}{2} + i14.134725...$ and $z_2 = \frac{1}{2} + i21.022040...$ and the complex conjugate of them. The corresponding contribution will be denoted by $r_1(x)$ (from z_1 and $\overline{z}_1$) and $r_2(x)$ (from z_2 and $\overline{z}_2$).

The contribution of the trivial zeros $z = -2n$ for every integers n, to the critical function is given by:

$$\psi_t(x;\alpha,\beta,\rho) = \frac{1}{\beta}\sum_{n=1}^{\infty} \frac{e^{-\frac{2n+\rho}{\beta}x}\Gamma(\frac{\alpha+2n}{\beta})}{\zeta'(-2n)}, \tag{1.9}$$

where a summation until $N = 20$ will be sufficient.

So, in our calculations we will set $\alpha = \frac{15}{2}$ and $\beta = 4$ in the above formulas, for any value of ρ we shall consider. The contribution ψ_t for ρ will be indicated with $g_\rho(x)$. Below we present the results of our numerical experiments performed using Maple 10, where as anticipated the maximum argument in the Möbius function present in the definition of the critical function (essentially c_k), is $N = 2000$. The fluctuations errors around 2000 will be specified in the Appendix.

2. Numerical experiments

In Fig. 1 we give the plot of the two functions $\psi(x; \frac{15}{2}, 4, \frac{1}{2}) - r_1(x) - r_2(x)$ and $g_{1/2}(x)$ up to $x = 30$ which shows a very good agreement. Notice that we have taken into account only the contribution of the first two nontrivial zeros in the Baez-Duarte asymptotic formula for the c_k. For the Riesz case ($\alpha = \beta = 2$), the contribution of the trivial zeros to the c_k have been treated by Maslanka using the Rice's integrals [8].

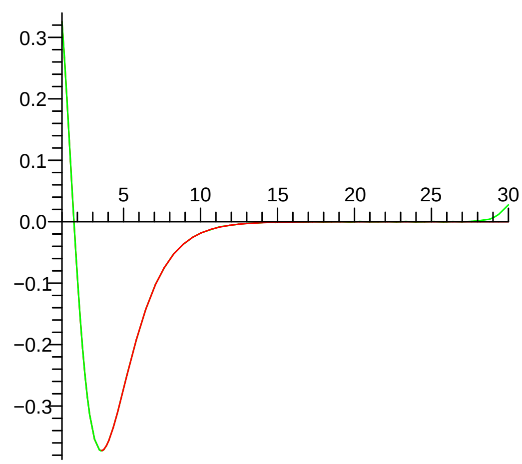

Figure 1. Plot of $\psi(x) - r_1(x) - r_2(x)$ [red] and $g_{1/2}(x)$ [green] up to $x = 30$

In Fig. 2 we present the the plot of the two functions $\psi(x; \frac{15}{2}, 4, \frac{1}{2}) - g_{1/2}(x)$ and $r_1(x) + r_2(x)$ up to $x = 30$ which shows not only a good agreement but also the oscillatory behaviour of the contribution of the first two nontrivial zeros.

In the next Fig. 3 we present the plots of some critical functions (ψ_ρ) corresponding to different values of ρ using (1.7) and this without any comparison with the Baez-Duarte asymptotic expansion considered above. It is to be noted that all functions ψ_ρ has the same zeros and we observe that there is a well marked evidence that for $\rho > \frac{1}{2}$ increasing to 1 the amplitudes decay while for $\rho < \frac{1}{2}$ the amplitudes grow. These functions have been indicated with ψ_1, $\psi_{7/8}$, $\psi_{3/4}$, $\psi_{5/8}$, $\psi_{1/2}$, $\psi_{3/8}$, $\psi_{3/10}$ respectively.

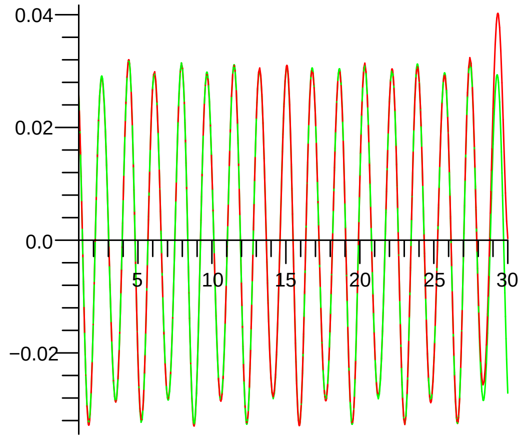

FIGURE 2. Plot of $\psi(x) - g_{1/2}(x)$ [red] and $r_1(x) + r_2(x)$ [green] up to $x = 30$

It should be said that $\psi_{3/8}$ and $\psi_{3/10}$, we have considered, have no relation with the representation of $\frac{1}{\zeta(s)}$ which is valid only for $\Re(s) > \frac{1}{2}$. The two functions help only to visualize that $\psi_{1/2}$ is the borderline for the critical functions decaying for $\Re(s) > \frac{1}{2}$ as suggested by our numerical experiments up to $x = 30$. It should also be added that from the duality relation (Riemann symmetry of the Zeta function), given by:

$$\frac{1}{\zeta(1-s)} = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \frac{1}{\zeta(s)}, \tag{2.1}$$

it follows that the right hand side of (2.1) ensures a representation of $\frac{1}{\zeta(s)}$ via the Pochammer's polynomials in the region $0 < \Re(s) < \frac{1}{2}$.

In Fig. 4 we present the result for a special case where we allow a slower decrease in the critical function (see addendum in the exponent of the critical function), which is the same as to say that we ask only for a slower decay of c_k, at $\rho = \frac{1}{2}$ i.e. of the type $c_k = \frac{A \log(k)}{k^{\frac{7}{4}}}$ for the case considered. This is not the same as to ask that RH is true or that RH is true with nontrivial zeros which are simple [3]. It is a case in between the two.

In this case the critical function (indicated with $\psi_{1/2+}$) is explicitly given by:

$$\psi_{1/2+}(x) = e^{\frac{7}{4}x - \log(x)} \sum_{n=1}^{2000} \frac{\mu(n)}{n^{\frac{15}{2}}} e^{-\frac{e^x}{n^4}}. \tag{2.2}$$

Here there is more evidence that the amplitude of the wave at $\rho = \frac{1}{2}$ is decreasing with $x = \log(k)$. The experiments of Fig. 3 give in any cases a stronger evidence:

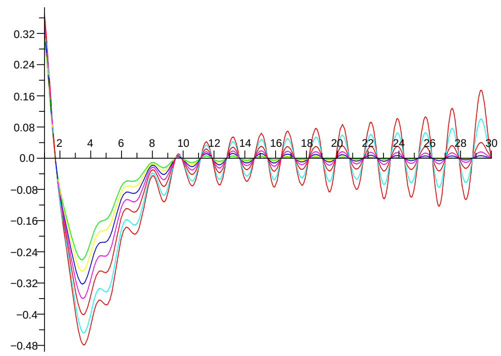

FIGURE 3. Plot of ψ_ρ for $\rho = 1, \frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{3}{10}$ up to $x = 30$, in order of increasing amplitudes

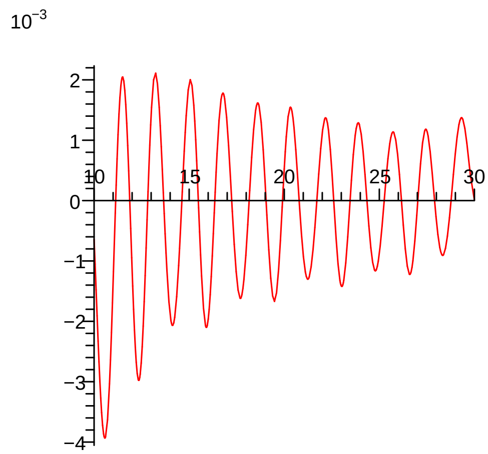

FIGURE 4. Plot of $\psi_{1/2+}$

that for $\rho > \frac{3}{4}$ the amplitudes of the waves are decaying, and thus are bounded in amplitude by a constant. This is a symptom of the absence of nontrivial zeros in the critical segment $\frac{3}{4} < \rho < 1$.

In the last experiment we set $\rho = \frac{3}{4}$ and compare $\psi_{3/4}$ with the asymptotic expression of Baez-Duarte: for the trivial zeros we set $\rho = \frac{3}{4}$ in the above formula, for the nontrivial zeros (the two we consider) we keep the same value of $\Im(z)$ but we

assume that their real part is $\Re(z) = \frac{3}{4}$. The plot in Fig. 5 of the function $\psi_{3/4}(x)$ and of $g_{3/4}(x)+r_1(x)+r_2(x)$ are clearly different: in $\psi_{3/4}$ there is the trace via the Möbius function of where the nontrivial zeros are located and thus the amplitude is decaying. In the second function, the two considered zeros are supposed to have $\Re(z) = \frac{3}{4}$ and the wave which appears seems to have a constant amplitude as in the case $\psi_{1/2}$ which of course would be sufficient to ensure the truth of the RH.

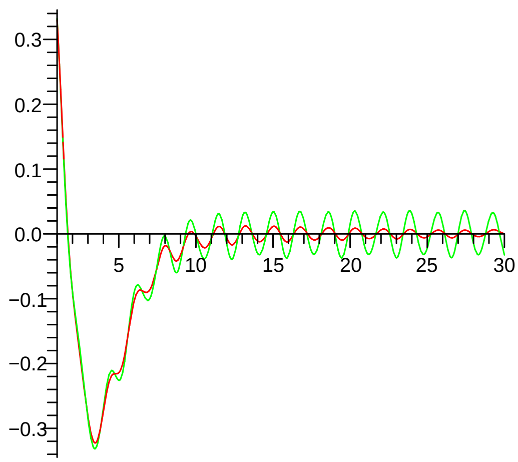

FIGURE 5. Plots of the functions $\psi_{3/4}(x)$ [red] and $g_{3/4}(x)+r_1(x)+r_2(x)$ [green]

In Appendix we analyse a (weak) stability property of our results obtained with $N = 2000$ in the Möbius function and give some indications why the waves for $\rho = \frac{3}{4}$ should be decaying, thus ensuring more credibility on the absence of zeros of the Riemann Zeta function in the segment $\frac{3}{4} < \rho < 1$.

3. Conclusions

In this work we have analyzed numerically the behaviour of the Riesz and Hardy-Littlewood wave (the critical function) in some details for the case $\alpha = \frac{15}{2}$ and $\beta = 4$ in the region up to about $k = 10'000$ milliard for various values of ρ in the critical segment $\frac{1}{2} < \rho < 1$. In the variable $x = \log(k)$ up to 14 oscillations have been detected whose amplitude has been compared with the one calculated with the expansion of Baez-Duarte using the trivial zeros and only the two lower nontrivial zeros. The agreement is satisfactory and the results give some indication in the direction to believe that at least for $\Re(s) = \rho > \frac{3}{4}$ there are no nontrivial zeros of $\zeta(s)$ since in the representation $\frac{1}{\zeta(s)}$ seems to remain bounded. In addition, we have given some evidence that a slow decay of c_k like $\frac{\log(k)}{k^{\frac{7}{4}}}$ in between to the decay $\frac{k^{\epsilon}}{k^{\frac{7}{4}}}$ (RH for the model) and $\frac{A}{k^{\frac{7}{4}}}$ (RH for the model with simple zeros) is possible in

the range $\log(k) < 30$. A further study in this direction but by means of two new representations of the Zeta function with coefficients b_k and d_k and other type of oscillations will be presented in the near future.

Appendix A.

We consider the critical function $\psi_{3/4}(x)$ obtained with $N = 2000$ (maximum argument in the Möbius function appearing in the Baez-Duarte definition of the c_k). We will suppose that the numerical results are given with good accuracy; we now ask: if we increase N from 2000 up to 10^6 in a ideal experiment, what will be the change of the critical function in the range $x < 30$?

$$\psi_{3/4}(x; N = 2000) = e^{\frac{27}{16}x} \sum_{n=1}^{2000} \frac{\mu(n)}{n^{\frac{15}{2}}} e^{-\frac{e^x}{n^4}}$$

$$\psi_{3/4}(x; N = 10^6) = e^{\frac{27}{16}x} \sum_{n=1}^{10^6} \frac{\mu(n)}{n^{\frac{15}{2}}} e^{-\frac{e^x}{n^4}}.$$

The difference Δ between the two functions is bounded ($|\mu(n)| \leq 1$) by:

$$\Delta \leq e^{\frac{27}{16}x} \sum_{n=2000}^{10^6} \frac{1}{n^{\frac{15}{2}}} e^{-\frac{e^x}{10^{24}}}.$$

If we ask that Δ will be smaller then say 10^{-6} time 0.015 which is about the value of the amplitude of the wave in the range $x \leq 30$, obtained with $N = 2000$, we have:

$$\Delta \leq e^{\frac{27}{16}x} e^{-\frac{e^x}{10^{24}}} (\zeta(\frac{15}{2}) - \zeta(\frac{15}{2}; N = 2000)) \leq 0.015 \cdot 10^{-6}.$$

The difference between the Zetas is estimated to:

$$\int_{2000}^{\infty} \frac{1}{x^{\frac{15}{2}}} dx = \frac{2}{13} 2000^{-\frac{13}{2}} = \frac{2}{65} 10^{-26}.$$

And the inequality takes the form:

$$\frac{27}{16}x - e^{\frac{x}{10^{24}}} + \log(\frac{2}{65}) - 26\log(10) + 6\log(10) - \log(0.015) \leq 0,$$

with the solution $x \leq 27$. Thus for $x \leq 27$, the amplitudes will change at most 10^{-6} time of its value 0.015. This shows some stability in the numerical experiments as N increases in a ideal experiment. Of course this is independent of how many zeros are employed in the Baez-Duarte estimation.

Finally, application of the crude inequality as above shows [5] that for larger values of x, with N sufficiently big, the numerical results of an ideal experiment using the Möbius function in c_k will give a value of the amplitude for example smaller then $\frac{1}{2}$ of 0.015 (still with $N = 2000$ at $x > 35$, with $N = 10^9$ at $x > 88$), indicating that the value of the amplitude becomes possibly smaller. This is an indication in the direction to believe that for $\rho > \frac{3}{4}$ there are no nontrivial zeros of the Zeta function.

To end up with the Appendix it should be remarked that in our experiments we have used the formula for the c_k [4], given by:

$$\hat{c}_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} e^{-\frac{k}{n^\beta}},$$

instead of the formula:

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k.$$

Again, as above, the crude inequality $|\mu(n)| \leq 1$ may be used to show that the difference between the two functions i.e. the fluctuations become smaller as k get bigger and depends on α and β. In fact they behave unconditionally as:

$$A \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+\beta}} \left(1 - \frac{1}{n^\beta}\right)^k \leq \frac{C}{k^{\frac{\alpha+\beta-1}{\beta}}}.$$

To see that, let $\Delta = |\hat{c}_k - c_k|$ then:

$$\Delta \leq \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^\alpha} \left(e^{-\frac{k}{n^\beta}} - \left(1 - \frac{1}{n^\beta}\right)^k\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \left(e^{-\frac{k}{n^\beta}} - \left(1 - \frac{1}{n^\beta}\right)^k\right),$$

since $e^{-\frac{k}{n^\beta}} \geq (1 - \frac{1}{n^\beta})^k$ and passing to the continuous variable x, the contribution of the second integral is given by [4]:

$$\int_1^\infty \frac{1}{x^\alpha} \left(1 - \frac{1}{x^\beta}\right)^k dx = \frac{1}{\beta} \frac{\Gamma(\frac{\alpha-1}{\beta})\Gamma(k+1)}{\Gamma(\frac{\alpha-1}{\beta} + k + 1)},$$

while the first is given by:

$$\int_1^\infty \frac{e^{-\frac{k}{x^\beta}}}{x^\alpha} dx = \frac{1}{\beta k^{\frac{\alpha-1}{\beta}}} \Gamma(\frac{\alpha-1}{\beta}).$$

At large k the fluctuation behaves like the difference, i.e. as:

$$\Delta \leq \frac{C}{k^{\frac{\alpha+\beta-1}{\beta}}}.$$

For the model under consideration the decay is as $\frac{C}{k^{\frac{21}{8}}}$ and is stronger then in the usual Riesz case ($\alpha = \beta = 2$) where an early more detailed calculation gives a decay like $\frac{C}{k^{\frac{3}{2}}}$ [6].

Finally it should be added that the general upper bound for Δ is related to the discrete derivative of the Baez-Duarte coefficients given by:

$$\begin{aligned} c_k - c_{k+1} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} \left(\left(1 - \frac{1}{n^\beta}\right)^k - \left(1 - \frac{1}{n^\beta}\right)^{k+1} \right) \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k \left(1 - 1 + \frac{1}{n^\beta}\right) = c_k(\alpha+\beta, \beta), \end{aligned}$$

which unconditionally are bounded by $\frac{C}{k^{\frac{\alpha+\beta-1}{\beta}}}$ as above [4].

In the same way

$$-\frac{d}{dk}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{\alpha}}e^{-\frac{k}{n^{\beta}}}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{\alpha+\beta}}e^{-\frac{k}{n^{\beta}}},$$

which gives the same decay since the function is equal to $c_k(\alpha+\beta,\beta)$ as above.

At large k we also have [4]:

$$c_k \approx \sum_{p=0}^{\infty}\frac{c_p k^p e^{-k}}{p!}$$

a Poisson like distribution for the coefficients c_k.

References

[1] S. Albeverio, C. Cebulla, Müntz formula and tero free regions for the Riemann Zeta function, 2006, Bull. Sci. Math.

[2] L. Baez-Duarte, A new necessary and sufficient condition for the Riemann Hypothesis, 2003, math.NT/0307215

[3] L. Baez-Duarte, A sequential Riesz-like criterion for the Riemann Hypothesis, 2005, International Journal of Mathematical Sciences, pp. 3527-3537

[4] S. Beltraminelli, D. Merlini, The criteria of Riesz, Hardy-Littlewood et al. for the Riemann Hypothesis revisited using similar functions, 2006, math.NT/0601138

[5] S. Beltraminelli, D. Merlini, Riemann Hypothesis: The Riesz-Hardy-Littlewood wave in the long wavelength region, 2006, math.NT/0605565

[6] J. Cislo, M. Wolf, Equivalence of Riesz and Baez-Duarte criterion for the Riemann Hypothesis, 2006, math.NT/0607782

[7] M. Coffey, On the coefficients of the Baez-Duarte criterion for the Riemann Hypothesis and their extensions, 2006, math-ph/0608050

[8] K. Maslanka, Baez-Duarte criterion for the Riemann Hypothesis and Rice's Integrals, 2006, math.NT/0603713

[9] M. Wolf, Evidence in favor of the Baez-Duarte criterion for the Riemann Hypothesis, 2006, math.NT/0605485

S. Beltraminelli, CERFIM, Research Center for Mathematics and Physics, PO Box 1132, 6600 Locarno, Switzerland

E-mail address: `stefano.beltraminelli@ti.ch`

D. Merlini, CERFIM, Research Center for Mathematics and Physics, PO Box 1132, 6600 Locarno, Switzerland

E-mail address: `merlini@cerfim.ch`